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Optimality conditions and duality for semi-infinite programming involving B-arcwise connected functions

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Abstract In this paper, a class of functions called B-arcwise connected (BCN) and strictly B-arcwise connected (STBCN) functions are introduced by relaxing definitions of arcwise connected function (CN) and B-vex function. The differential properties of B-arcwise connected function (BCN) are studied. Their two extreme properties are proved. The necessary and sufficient optimality conditions are obtained for the nondifferentiable nonlinear semi-infinite programming involving B-arcwise connected (BCN) and strictly B-arcwise connected (STBCN) functions. Mond-Weir type duality results have also been established.

Keywords B-arcwise connected function (BCN) · Semi-infinite programming · Optimality condition · Duality

Mathematics Subject Classification (2000) 90C34 · 90C46 · 26B25

1 Introduction

Avriel and Zang [2] extended the concept of convex function by defining arcwise connected (CN), Q-connected (QCN), strongly Q-connected (SQCN), and P-connected (PCN) functions. Bector and Singh [3] also extended the concept of convex function by defining B-vex function. Mehra and Bhatia [16] studied optimality conditions and duality results for minmax problems involving arcwise connected and generalized arcwise connected functions. Bhatia and Mehra [4] investigated some properties of arcwise connected functions in terms of their derivatives. The necessary and sufficient optimality conditions are presented, and Mond-Weir type duality results are also proved. Davar and Mehra [7] obtained optimality conditions and

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duality results for fractional programming problems involving arcwise connected functions and their generalizations.

Recently, many scholars have been making deeper research for semi-infinite programming (SIP). Many good results have obtained especially in semi-infinite linear programming (LSIP) and semi-infinite convex programming (convex SIP) [9,17,18]. For instance, the duality and dual gaps for LSIP were studied by Charnes et al. [6], Goberna and Lóṕez [8], Liu [14], and Karney [11], respectively. Lóṕez and Vercher [15] presented optimality conditions for nondifferentiable convex SIP. Borwein [5] has proven a semi-infinite quasi-convex program with certain regularity conditions possessing finite constrained subprogram with the same optimal value. Rüćkmann and Shapiro [19] have given first-order Optimality conditions in generalized SIP. The duality for convex SIP is investigated by Jeroslow [10], Karney and Morley [12] and Li [13], respectively. The optimality conditions and duality results are obtained and established by Zhang [21] for arcwise connected semi-infinite programming problems.

The paper mainly includes two aspects. One is to extend arcwise connected (CN) and B-vex functions. The other is to give optimality conditions and obtain duality results for semi-infinite programming with the new generalized convex functions presenting in the first aspect. This paper is organized as follows. Section 2 introduces B-arcwise connected (BCN) and strictly B-arcwise connected (BSTCN) functions, and presents some properties. In Sect. 3, we give some optimality conditions for semi-infinite programming problem involving B-arcwise connected (BCN) and strictly B-arcwise connected (BSTCN) functions. In Sect. 4, the Mond-Weir type duality results are obtained for the nonlinear semi-infinite programming problem involving these generalized convex functions. In Sect. 5, conclusions and outlook are given.

2 B-arcwise connected functions

In this section, we shall focus our attention on certain extensions of known families of generalized convex functions. Let C be a nonempty open subset of $\mathbb{R}^n$, let $b(x, y, \theta)$: $C \times C \times [0, 1] \longrightarrow \mathbb{R}_+, 0 \leq \theta \leq 1, 0 \leq \theta b(x, y, \theta) \leq 1$.

Definition 2.1 [1,2] A set $C \subseteq \mathbb{R}^n$ is said to be an arcwise connected (AC) set if, for every $x^1 \in C, x^2 \in C$, there exists a continuous vector-valued function $H_{x^1,x^2} : [0, 1] \longrightarrow C$, called an arc, such that

$$H_{x^1,x^2}(0) = x^1, \quad H_{x^1,x^2}(1) = x^2.$$

Definition 2.2 [2] Let f be a real-valued function defined on an AC set $C \subset \mathbb{R}^n$. Then

(i) f is said to be an arcwise connected function (CN) if, for every $x^1 \in C, x^2 \in C$, there exists an arc H_{x^1,x^2} such that

$$f(H_{x^1,x^2}(\theta)) \leq (1-\theta) f(x^1) + \theta f(x^2), \quad \text{for } 0 \leq \theta \leq 1.$$

(ii) f is said to be a strictly arcwise connected function (STCN) if, for every $x^1 \in C, x^2 \in C, x^1 \neq x^2$, there exists an arc H_{x^1,x^2} such that

$$f(H_{x^1x^2}(\theta)) < (1-\theta) f(x^1) + \theta f(x^2), \quad \text{for } 0 < \theta < 1.$$

Definition 2.3 [3] Let f be a real-valued function defined on a convex set $C \subset \mathbb{R}^n$. Then

(i) f is said to be an B-vex function if, for every $x^1 \in C, x^2 \in C$, there exists a real function $b(x^1, x^2, \theta)$ such that

$$f(\theta x^1 + (1-\theta)x^2) \leq \theta b(x^1, x^2, \theta) f(x^1) + (1 - \theta b(x^1, x^2, \theta)) f(x^2), \quad \text{for } 0 \leq \theta \leq 1, 0 \leq \theta b \leq 1.$$

(ii) f is said to be a strictly B-vex function if, for every $x^1 \in C, x^2 \in C, x^1 \neq x^2$, there exists a real function $b(x^1, x^2, \theta)$ such that

$$f(\theta x^1 + (1-\theta)x^2) < \theta b(x^1, x^2, \theta) f(x^1) + (1 - \theta b(x^1, x^2, \theta)) f(x^2), \quad \text{for } 0<\theta<1.$$

f is said to be a concave CN (COCN) and concave STCN (COSTCN) function, respectively, if $-f$ is a CN and STCN function, respectively. f is said to be a B-cave and strictly B-cave function, respectively, if $-f$ is a B-vex and strictly B-vex function, respectively.

We introduce the following concepts of B-arcwise connected (BCN) and strictly B-arcwise connected functions (STBCN) based no the definitions of arcwise connected (CN) and B-vex functions.

Definition 2.4 Let f be a real-valued function defined on an AC set $C \subset \mathbb{R}^n$. Then

(i) f is said to be a B-arcwise connected function (BCN) if, for every $x^1 \in C$, $x^2 \in C$, there exist an arc H_{x^1,x^2} and a real function $b(x^1, x^2, \theta)$ such that

$$f(H_{x^1,x^2}(\theta)) \leq (1 - \theta b(x^1, x^2, \theta)) f(x^1) + \theta b(x^1, x^2, \theta) f(x^2), \quad \text{for } 0 \leq \theta \leq 1, 0 \leq \theta b \leq 1.$$

(ii) f is said to be a strictly B-arcwise connected function (STBCN) if, for every $x^1 \in C$, $x^2 \in C, x^1 \neq x^2$, there exist an arc H_{x^1,x^2} and a real function b such that

$$f(H_{x^1x^2}(\theta)) < (1 - \theta b(x^1, x^2, \theta)) f(x^1) + \theta b(x^1, x^2, \theta) f(x^2), \quad \text{for } 0 < \theta < 1, 0 < \theta b < 1.$$

f is said to be a concave BCN (COBCN) and concave STBCN (COSTBCN) function, respectively, if $-f$ is a BCN and STBCN function, respectively.

Definition 2.5 [2,20] Let f be a real-valued function defined on an AC set $C \subset \mathbb{R}^n$. Let $x^1 \in C, x^2 \in C$ and H_{x^1,x^2} be the arc connecting x^1 and x^2 in C. The function f is said to possess a right derivative or right differential with respect to an arc H_{x^1,x^2} at $\theta = 0$ if

$$\lim_{\theta \to 0^+} \frac{f(H_{x^1,x^2}(\theta)) - f(x^1)}{\theta}$$

exists. This limit is denoted by $f^+(H_{x^1,x^2}(0))$. If

$$\lim_{\theta \to 0^+} \frac{H_{x^1,x^2}(\theta) - x^1}{\theta}$$

exists and we denote it by $H^+_{x^1,x^2}(0)$, then vector $H^+_{x^1,x^2}(0)$ is called directional derivative of H_{x^1,x^2} at $\theta = 0$.

The right derivative or right differential of f can be written as

$$f^{+}(H_{x^*,x}(0)) = H^{+}_{x^*,x}(0)\nabla f(x^*)^T,$$

whenever $f^{+}(H_{x^1,x^2}(0))$ and $H^{+}_{x^1,x^2}(0)$ exist.

Definition 2.6 A real function f defined on an AC set $C \subseteq R^n$ is said to be directional differentially B-arcwise connected (DBCN) [directional differentially strictly B-arcwise connected (DSTBCN)] if for every $x^1 \in C, x^2 \in C$ [if for every $x^1 \in C, x^2 \in C, x^1 \neq x^2$], there exist an arc H_{x^1,x^2} in C and a real function b such that the following two conditions are satisfied:

(i) f is BCN function with respect to b and H_{x^1,x^2}, and $\bar{b}(x^1, x^2) = \lim_{\theta \to 0^+} b(x^1, x^2, \theta)$.
(ii) f possess a right derivative or right differential $f^{+}(H_{x^1,x^2}(0))$, with respect to an arc H_{x^1,x^2} at $\theta = 0$.

Remark 2.1 The sum of two BCN is not necessarily BCN, unless an extra restriction is required that they be BCN with respect to the same arc and function b for each pair of points.

Remark 2.2 (i) Every CN function is BCN function, however, the converse is not necessarily true.

(ii) Every STCN function is STBCN function, however, the converse is not necessarily true.

(iii) Every B-vex function is BCN function, however, the converse is not necessarily true.

(iv) Every strictly B-vex function is STBCN function, however, the converse is not necessarily true.

Below, we give an example.

Example 2.1 Define f as

$$f(x) = \begin{cases} (x_1x_2)^2, & \text{if } x_1x_2 \leq 4, \\ 16, & \text{otherwise.} \end{cases}$$

The level set of this function is denoted by $S(f, \alpha)(\forall \alpha \in \mathbb{R})$ and level curves of the function are shown in Fig. 1. It is clear that this function is not quasiconvex, convex, or concave. For every $x^1 \in \mathbb{R}^2, x^2 \in \mathbb{R}^2$, we define arc as

$$H_{x^1,x^2}(\theta) = \begin{cases} (1-2\theta)x^1, & \text{if } \ 0 \leq \theta \leq \dfrac{1}{2} \\ (2\theta - 1)x^2, & \text{if } \ \dfrac{1}{2} \leq \theta \leq 1 \end{cases}$$

and function $b : C \times C \times [0, 1] \longrightarrow \mathbb{R}_+$ as

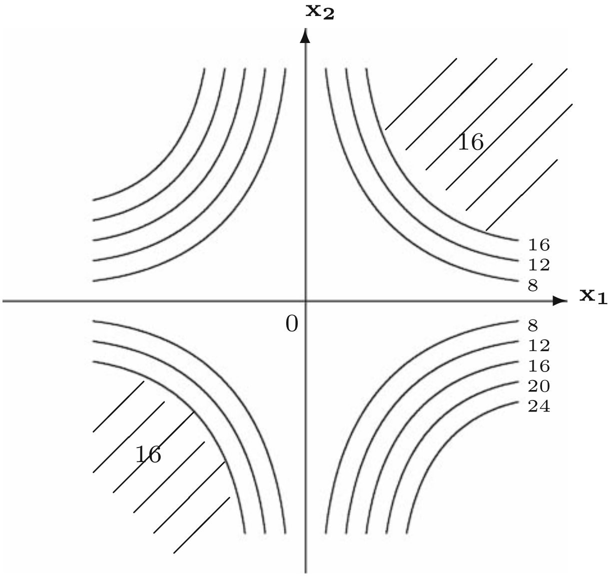

Fig. 1 A BCN function

$$b(x^1, x^2, \theta) = \begin{cases} 4 - 4\theta, & \text{if } 0 \le \theta \le \frac{1}{2} \text{ and } x_1^1 x_2^1 \le 4, x_1^2 x_2^2 \le 4 \\ (2\theta - 1)^2/\theta, & \text{if } \frac{1}{2} < \theta \le 1 \text{ and } x_1^1 x_2^1 \le 4, x_1^2 x_2^2 \le 4 \\ 0, & \text{if } 0 \le \theta \le \frac{1}{2} \text{ and } x_1^2 x_2^2 > 4 \quad \text{or } x_1^1 x_2^1 > 4, x_1^2 x_2^2 > 4 \\ 1/\theta, & \text{if } \frac{1}{2} < \theta \le 1 \text{ and } x_1^2 x_2^2 > 4. \end{cases}$$

Then, we can prove that it is a BCN function on $\mathbb{R}^2$. However, we have

$$f(H_{x^1,x^2}(\theta)) > (1-\theta)f(x^1) + \theta f(x^2), \quad \text{for } \theta = 1/4, x^1 = (2, 8), x^2 = (0, 0)$$

and

$$f(H_{x^1,x^2}(\theta)) < (1-\theta)f(x^1) + \theta f(x^2), \quad \text{for } \theta = 1/4, x^1 = (2, 0), x^2 = (4, 1),$$

therefore, f is neither a CN nor a $COCN$ function. Again, we obtain

$$f(\theta x^1 + (1-\theta)x^2) > \theta b(x^1, x^2, \theta) f(x^1) + (1 - \theta b(x^1, x^2, \theta)) f(x^2), \text{for } \theta = 1/4, x^1 = (0, 0), x^2 = (1, 4)$$

and

$$f(\theta x^1 + (1-\theta)x^2) < \theta b(x^1, x^2, \theta) f(x^1) + (1 - \theta b(x^1, x^2, \theta)) f(x^2), \quad \text{for } \theta = 1/4, x^1 = (4, 1), x^2 = (2, 1),$$

hence, f is neither a $B-$vex nor a $B-$cave function.

We now give the differential and extreme properties of BCN and STBCN functions. Their other basic properties will be studied in future paper.

Theorem 2.1 *Assume that f is a real-valued DBCN function on C, then*

$$f^+(H_{x^1,x^2}(0)) \leq \bar{b}(x^1, x^2)[f(x^2) - f(x^1)]$$

for $0 < \theta < 1$, *where* $\bar{b}(x^1, x^2) = \lim_{\theta \to 0^+} b(x^1, x^2, \theta)$.

Proof Let f be a DBCN function, then, for every $x^1 \in C, x^2 \in C$, there exist an arc $H_{x^1,x^2}(\theta)$ and a real function $b(x^1, x^2, \theta)$ such that

$$f(H_{x^1,x^2}(\theta)) \leq (1 - \theta b(x^1, x^2, \theta)) f(x^1) + \theta b(x^1, x^2, \theta) f(x^2)$$

for $0 < \theta < 1$. It follows that

$$\frac{f(H_{x^1,x^2}(\theta)) - f(x^1)}{\theta} \leq b(x^1, x^2, \theta)[f(x^2) - f(x^1)].$$

Let $\theta \to 0^+$, we have

$$f^+(H_{x^1,x^2}(0)) \leq \bar{b}(x^1, x^2)[f(x^2) - f(x^1)].$$

□

Corollary 2.1 *Assume that f is a real-valued DSTBCN function on C, then*

$$f^+(H_{x^1,x^2}(0)) \leq \bar{b}(x^1, x^2)[f(x^2) - f(x^1)]$$

for $0 < \theta < 1$, *where* $\bar{b}(x^1, x^2) = \lim_{\theta \to 0^+} b(x^1, x^2, \theta)$.

Theorem 2.2 *Assume that f is a real-valued DBCN function on AC set $C \subset R^n$, and let $b(x^*, x) = \lim_{\theta \to 0^+} b(x^*, x, \theta) > 0$. If $x^* \in C$ is a point where $\nabla f(x^*) = 0$, then x^* is a global minimum point of f on C.*

Proof Let f be a DBCN, and Suppose that $\nabla f(x^*) = 0$, then for every $x \in C$, arc $H_{x^*,x}$ and a real function $b(x^*, x)$, we have

$$b(x^*, x)[f(x) - f(x^*)] \geq f^+(H_{x^*,x}(0)) = H^+_{x^*,x}(0)\nabla f(x^*)^T = 0.$$

From that it follows

$$b(x^*, x) f(x) \geq b(x^*, x) f(x^*).$$

Since $b(x^*, x) > 0$, thus $f(x) \geq f(x^*)$, then x^* is a global minimum point of f on C. □

Theorem 2.3 *Assume that f is a real-valued DSTBCN function on AC set $C \subset R^n$, and let $b(x^*, x) = \lim_{\theta \to 0^+} b(x^*, x, \theta) > 0$. If $x^* \in C$ is a point where $\nabla f(x^*) = 0$, then x^* is a unique global minimum point of f on C.*

Proof Similarly to proof of Theorem 2.2 we can show that x^* is a global minimum point of f on C.

Now we prove uniqueness. If x^0 is other global minimum point of f on C, then $f(x^0) = f(x^*)$, $x^0 \neq x^*$. Since f is STBCN, then there exist an arc $H_{x^*,x^0}(\theta)$ and a real-valued function $b(x^*, x^0, \theta)$ such that

$$f(H_{x^*,x^0}(\theta)) < (1 - \theta b(x^*, x^0, \theta)) f(x^*) + \theta b(x^*, x^0, \theta) f(x^0) = f(x^*), \quad \text{for } 0 < \theta < 1,$$

this contradicts that x^* is a global minimum of f on C. □

3 Optimality conditions

Consider the following nonlinear semi-infinite programming problem (P):

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } g(x,u) \le 0 (u \in U) \\ &\qquad\qquad x \in X, \end{aligned}$$

where $X \subset \mathbb{R}^n$ is a nonempty open AC set, $U \subset \mathbb{R}^m$ is an infinite countable set, $f : X \longrightarrow \mathbb{R}$, $g : X \times U \longrightarrow \mathbb{R}$ are real-valued functions, the right derivatives of the functions $f(x)$ and $g(x,u)$ $(u \in U)$ with respect to an arc H_{x^1,x^2} at $\theta = 0$ exist for every $x^1 \in X, x^2 \in X$. The feasible region of problem (P) is denoted by $X^0 = \{x \in X | g(x,u) \le 0, u \in U\}$. Let $\Delta = \{i | g(x,u^i) \le 0, x \in X, u^i \in U\}$ and $\Lambda = \{\lambda = (\lambda_i)_{i\in\Delta} \mid$ only finitely many $\lambda_i \ne 0\}$. We define $\tilde{U}(x^*) := \{u^i | g(x^*,u^i) = 0, u^i \in U\}$, the set of u for which our constrait is active. Let $I(x^*) := \{i | g(x^*,u^i) = 0\} = \{i | u^i \in \tilde{U}\}$ and $J(x^*) := \{i | g(x^*,u^i) < 0\}$.

Below, we study optimality of problem (P). The theorems of the optimality for problem (P) are given and proven as follows:

Lemma 3.1 *Let $g(x,u^i)(i = 1,2,\ldots)$ be a real-valued BCN function with respect to x defined on AC set $X \times U \subset \mathbb{R}^{n+m}$. Then, exactly one of the following systems is solvable:*

- (i) *there exists $x \in X$ such that $g(x,u^i) < 0$ $(i = 1,2,\ldots)$.*
- (ii) *there exists $\lambda \in \Lambda, \lambda \ge 0$ such that $\sum_{i\in\Delta} \lambda_i g(x,u^i) \ge 0$, for all $x \in X$.*

Proof If the systems (i) has a solution, then, for every $x^1 \in X, x^2 \in X$, there exist an arc H_{x^1,x^2} and a real function b such that

$$g(H_{x^1,x^2}(\theta),u^i) \le (1-\theta b(x^1,x^2,\theta))g(x^1,u^i) + \theta b(x^1,x^2,\theta)g(x^2,u^i) < 0 \quad (3.1)$$

for $0 \le \theta b \le 1$, hence $H_{x^1,x^2}(\theta) \in X$. For any $\lambda \in \Lambda, \lambda \ge 0$ and by (3.1), we obtain

$$\sum_{i\in\Delta} \lambda_i g(H_{x^1,x^2}(\theta),u^i) < 0. \quad (3.2)$$

If the systems (ii) has also a solution, then there exists $\lambda \in \Lambda, \lambda \ge 0$ such that $\sum_{i\in\Delta} \lambda_i g(H_{x^1,x^2}(\theta),u^i) \ge 0$ for the arc $H_{x^1,x^2}(\theta) \in X$, this contradicting (3.2), hence (ii) has no solution.

Similarly, we can show that if the systems (ii) has a solution, then systems (i) has no solution. □

Theorem 3.1 *(Fritz-John Type Necessary Optimality Condition) Assume that x^* is an optimal solution of (P). If $f^+(H_{x^*,x}(0))$ and $g^+(H_{x*,x}(0),u^{*i})(i \in I(x^*))$ are DBCN functions of x, $g(x,u^i)(i \in J(x^*))$ is continuous at x^* with AC set X. Then there exist $\lambda_0^* \in \mathbb{R}, \lambda^* \in \Lambda$ such that*

$$\lambda_0^* f^+(H_{x^*,x}(0)) + \sum_{i\in\Delta} \lambda_i^* g^+(H_{x^*,x}(0),u^i) \ge 0, \quad \textit{for all } x \in X, \quad (3.3)$$

$$\sum_{i\in\Delta} \lambda_i^* g(x^*,u^i) = 0, \quad (3.4)$$

$$\lambda_0^* \ge 0, \quad \lambda^* \ge 0. \quad (3.5)$$

Proof First we assert that the system

$$f^{+}(H_{x^*,x}(0)) < 0, \tag{3.6}$$
$$g^{+}(H_{x*,x}(0), u^i) < 0, \quad i \in \Delta \tag{3.7}$$

has no solution x in X.

If possible let $x \in X$ be a solution of system (3.6) and (3.7). Since right differentials of $f(x)$ and $g(x, u^i)(i \in I(x^*))$ at x^* exist with respect to the $H_{x*,x}$, therefore

$$f(H_{x^*,x}(\theta)) = f(x^*) + \theta f^{+}(H_{x^*,x}(0)) + \theta\alpha(\theta) \tag{3.8}$$

and

$$g(H_{x^*,x}(\theta), u^i) = g(x^*, u^i) + \theta g^{+}(H_{x^*,x}(0), u^i) + \theta\alpha_i(\theta), \tag{3.9}$$

where

$$\lim_{\theta\to 0^+} \alpha(\theta) = 0, \quad \lim_{\theta\to 0^+} \alpha_i(\theta) = 0, \quad i \in I(x^*). \tag{3.10}$$

Using (3.6), (3.7) and (3.10), we obtain, for small enough θ, $0 < \theta < \bar{\theta} < 1$

$$f^{+}(H_{x^*,x}(0)) + \alpha(\theta) < 0, \tag{3.11}$$
$$g^{+}(H_{x^*,x}(0), u^i) + \theta\alpha_i(\theta) < 0, i \in I(x^*). \tag{3.12}$$

Hence, by relations (3.8) and (3.9), we have, for $0 < \theta < \bar{\theta}$

$$f(H_{x^*,x}(\theta)) < f(x^*) \tag{3.13}$$

and

$$g(H_{x^*,x}(\theta), u^i) < g(x^*, u^i), i \in I(x^*). \tag{3.14}$$

Now, since $g(H_{x^*,x}(\theta), u^i)(i \in J(x^*))$ is continuous at x^* and $H_{x^*,x}(\theta)$ is also a continuous function of θ, therefore

$$\lim_{\theta\to 0^+} g(H_{x^*,x}(\theta), u^i) = g(x^*, u^i) < 0$$

which implies that there exist θ_i^*, $0 < \theta_i^* < 1 (i \in J(x^*))$, such that

$$g(H_{x^*,x}(\theta), u^i) < 0, \quad \text{for } 0 < \theta < \theta_i^*. \tag{3.15}$$

Let $\theta^* = min(\bar{\theta}, \theta_i^*)$. Thus using (3.13), (3.14) and (3.15) we get for $0 < \theta < \theta^*$, $H_{x^*,x}(\theta) \in X^0 \subset X$ and $f(H_{x^*,x}(\theta)) < f(x^*)$ which is a contradiction as x^* is an optimal solution of (P). Hence, the system (3.6) and (3.7) has no solution $x \in X$.

Since $f^{+}(H_{x^*,x}(0))$ and $g^{+}(H_{x*,x}(0), u^i)(i \in I(x^*))$ are BCN functions of x, therefore by Lemma 3.1, there exist $\lambda_0^* \in \mathbb{R}$, $(\lambda_i^*)_{i\in I(x^*)}$, $\lambda_i^* \in \mathbb{R}$, only finitely many nonzero, such that the following conditions hold:

$$\lambda_0^* f^{+}(H_{x^*,x}(0)) + \sum_{i\in I(x^*)} \lambda_i^* g^{+}(H_{x^*,x}(0), u^i) \geq 0,$$
$$\sum_{i\in\Delta} \lambda_i^* g(x^*, u^i) = 0,$$
$$\lambda_0^* \geq 0, \quad \lambda_i^* \geq 0, i \in I(x^*).$$

Again letting $\lambda_i^* = 0$ $(i \in J(x^*))$, then the conclusion of the theorem holds. □

Theorem 3.2 *(Fritz-John Type Necessary Optimality Condition) Assume that x^* is an optimal solution of (P). If $f(x)$ and $g(x, u^i)(i \in \Delta)$ are DBCN functions of x with respect to the same arc and function b. Then for $u^i \in U$, there exist $\lambda_0^* \in \mathbb{R}$, $\lambda^* \in \Lambda$ such that* (3.3)–(3.5) *hold at x^*.*

Proof Since x^* is an optimal solution of (P), hence

$$f(x) \geq f(x^*), \quad \forall x \in X^0 \subset X,$$

then the system

$$\begin{aligned} f(x) - f(x^*) &< 0 \\ g(x, u^i) &< 0 \end{aligned}$$

has no solution.

We now prove $F(x, u^i) = (f(x) - f(x^*), g(x, u^i))$ is a BCN function of x for every $u^i \in U$. Since $f(x)$ and $g(x, u^i)(i \in \Delta)$ are DBCN functions of x with respect to the same arc and function b for $x^1, x^2 \in X$, therefore there exist an arc $H_{x^1,x^2}(\theta)$ and a real function $b(x^1, x^2, \theta)$ such that

$$\begin{aligned} F(H_{x^1,x^2}(\theta), u^i) &= (f(H_{x^1,x^2}(\theta)) - f(x^*), \quad g(H_{x^1,x^2}(\theta), u^i)) \\ &\leq ((1 - \theta b(x^1, x^2, \theta)) f(x^1) + \theta b(x^1, x^2, \theta) f(x^2) \\ &\quad - f(x^*), (1 - \theta b(x^1, x^2, \theta)) g(x^1, u^i) \\ &\quad + \theta b(x^1, x^2, \theta) g(x^2, u^i)) \\ &= ((1 - \theta b(x^1, x^2, \theta))(f(x^1) - f(x^*)) + \theta b(x^1, x^2, \theta)(f(x^2) - f(x^*)), \\ &\quad (1 - \theta b(x^1, x^2, \theta)) g(x^1, u^i) + \theta b(x^1, x^2, \theta) g(x^2, u^i)) \\ &= (1 - \theta b(x^1, x^2, \theta))(f(x^1) - f(x^*), g(x^1, u^i)) \\ &\quad + \theta b(x^1, x^2, \theta)(f(x^2) - f(x^*), g(x^2, u^i)). \end{aligned}$$

Hence, $F(x, u^i) = (f(x) - f(x^*), g(x, u^i))(i \in \Delta)$ is a BCN function. By Lemma 3.1, there exist $\lambda_0^* \in \mathbb{R}, \lambda_0^* \geq 0, \lambda^* \in \Lambda, \lambda^* \geq 0$ such that

$$\lambda_0^*(f(x) - f(x^*)) + \sum_{i \in \Delta} \lambda_i^* g(x, u^i) \geq 0, \quad \text{for all } x \in X. \tag{3.16}$$

Taking $x = x^*$, we get $\sum_{i \in \Delta} \lambda_i^* g(x^*, u^i) \geq 0$. Since $\lambda_0 \geq 0$, $\lambda^* \geq 0$ and x^* is feasible for (P), we have $\sum_{i \in \Delta} \lambda_i^* g(x^*, u^i) \leq 0$. Hence $\sum_{i \in \Delta} \lambda_i^* g(x^*, u^i) = 0$.

Since X is an AC set, thus $H_{x^*,x}(\theta) \in X$, for all $x \in X^0, 0 < \theta < 1$, by (3.16) and $\sum_{i \in \Delta} \lambda_i^* g(x^*, u^i) = 0$, we obtain

$$\lambda_0^*(f(H_{x^*,x}(\theta)) - f(x^*)) + \sum_{i \in \Delta} \lambda_i^*(g(H_{x^*,x}(\theta), u^i) - g(x^*, u^i)) \geq 0.$$

Dividing by $\theta > 0$ and then letting $\theta \to 0^+$, we obtain

$$\lambda_0^* f^+(H_{x^*,x}(0)) + \sum_{i \in \Delta} \lambda_i^* g^+(H_{x^*,x}(0), u^i) \geq 0, \quad \text{for all } x \in X,$$

i.e., the conclusion of the theorem holds true. □

Theorem 3.3 *(Karush-Kuhn-Tucker Type Necessary Optimality Condition) Assume that x^* is an optimal solution of (P) and let $f(x), g(x, u^i)(i \in \Delta)$ be DBCN functions of x with respect to the same arc $H_{x^*,x}(\theta)$ and $b(x^*, x) = \lim_{\theta\to 0_+} b(x^*, x, \theta) > 0$. If there exists $\hat{x} \in X^0$ such that $g(\hat{x}, u^i) < 0$, then for $u^i \in U$, there exist $\lambda_0^* \in \mathbb{R}$, $\lambda_0^* > 0$, $\lambda^* \in \Lambda$ such that (3.3)–(3.5) hold at x^*.*

Proof By theorem 3.2, there exist $\lambda_0^* \geq 0$ and $\lambda^* \in \Lambda$ such that (3.3)–(3.5) hold.

Now, suppose that $\lambda_0^* = 0$. Then (3.3)–(3.5) reduce to

$$\sum_{i\in\Delta} \lambda_i^* g^+(H_{x^*,x}(0), u^i) \geq 0, \quad \text{for all} \quad x \in X, \tag{3.17}$$

$$\sum_{i\in\Delta} \lambda_i^* g(x^*, u^i) = 0, \tag{3.18}$$

$$\lambda^* > 0. \tag{3.19}$$

Since $g(x, u^i)$ is a DBCN function, we have

$$g^+(H_{x^*,\hat{x}}(0), u^i) \leq b(x^*, \hat{x})[g(\hat{x}, u^i) - g(x^*, u^i)]. \tag{3.20}$$

Using (3.19) and (3.20), we obtain

$$\sum_{i\in\Delta} \lambda_i^* g^+(H_{x^*,\hat{x}}(0), u^i) \leq b(x^*, \hat{x}) \sum_{i\in\Delta} \lambda_i^* [g(\hat{x}, u^i) - g(x^*, u^i)]. \tag{3.21}$$

It follows from (3.17), (3.18) and (3.21) that

$$b(x^*, \hat{x}) \sum_{i\in\Delta} \lambda_i^* g(\hat{x}, u^i) \geq 0. \tag{3.22}$$

But since $b(x^*, \hat{x}) > 0$, thus (3.22) contradicts the facts that

$$\sum_{i\in\Delta} \lambda_i^* g(\hat{x}, u^i) < 0 \quad \text{and } \lambda^* \geq 0.$$

hence, $\lambda_0^* > 0$. □

Theorem 3.4 *Assume that $f(x)$ is a DB_0CN function and $g(x, u^i)(i \in I(x^*))$ is a DB_iCN function at x^* with respect to the same arc $H_{x^*,x}(\theta)$ and*

$$b_0(x^*, x) = \lim_{\theta\to 0_+} b_0(x^*, x, \theta) > 0, \quad b_i(x^*, x) = \lim_{\theta\to 0_+} b_i(x^*, x, \theta), 0 \leq \theta \leq 1.$$

If there exist $\lambda_0^* \in \mathbb{R}$, $\lambda_0^* > 0$, $\lambda^* \in \Lambda$, for all $x \in X^0$ and any $u^i \in U$ such that (3.3)-(3.5) hold at x^*, then x^* is an optimal solution of (P).

Proof If x^* is not an optimal solution of (P), then there exists $\bar{x} \in X^0$ such that

$$f(\bar{x}) < f(x^*). \tag{3.23}$$

Since $f(x)$ is a $\mathrm{DB_0CN}$ function, $g(x, u^i)(i \in I(x^*))$ is a $DB_i\mathrm{CN}$ function, and by Theorem 2.1 and (3.23), there exists an arc $H_{x^*,\bar{x}}(\theta), 0 \leq \theta \leq 1$, for $b_0(x^*, \bar{x}) > 0$, $b_i(x^*, \bar{x}) \geq 0$, we obtain

$$f^+(H_{x^*,\bar{x}}(0)) \leq b_0(x^*, \bar{x})[f(\bar{x}) - f(x^*)] < 0 \tag{3.24}$$

and

$$g^{+}(H_{x^*,\bar{x}}(0), u^i) \leq b_i(x^*, \bar{x})[g(\bar{x}, u^i) - g(x^*, u^i)], \quad i \in I(x^*). \tag{3.25}$$

For $\lambda_0^* > 0, \lambda^* \in \Lambda, \lambda^* \geq 0 \quad (\lambda_i^* = 0$ when $i \in J(x^*))$, from (3.24) and (3.25), we have

$$\lambda_0^* f^{+}(H_{x^*,\bar{x}}(0)) + \sum_{i\in\Delta} \lambda_i^* g^{+}(H_{x^*,\bar{x}}(0), u^i) < \sum_{i\in\Delta} \lambda_i^* b_i(x^*, \bar{x})[g(\bar{x}, u^i) - g(x^*, u^i)],$$

by (3.4), it follows that

$$\lambda_0^* f^{+}(H_{x^*,\bar{x}}(0)) + \sum_{i\in\Delta} \lambda_i^* g^{+}(H_{x^*,\bar{x}}(0), u^i) < 0,$$

which contradicts to (3.3). Hence x^* is an optimal solution of (P). □

The proof of the following theorem follows on the lines of Theorem 3.4; therefore, we state it without proof.

Theorem 3.5 *Assume that $f(x)$ is a DSTB$_0$CN function and let $g(x, u^i)(i \in I(x^*))$ be a DSTB$_i$CN or DB$_i$CN function at x^* with respect to the same arc $H_{x^*,x}(\theta)$ and*

$$b_0(x^*, x) = \lim_{\theta\to 0_+} b_0(x^*, x, \theta) > 0, \quad b_i(x^*, x) = \lim_{\theta\to 0_+} b_i(x^*, x, \theta), 0 < \theta < 1.$$

If there exist $\lambda_0^* \in \mathbb{R}, \lambda_0^* > 0, \lambda^* \in \Lambda$, for all $x \in X^0$ and any $u^i \in U$ such that (3.3)–(3.5) hold at x^*, then x^* is an optimal solution of (P).

Example 3.1 Let $X = \{(x_1, x_2)|x_1^2 + x_2^2 \geq 1, x_1 > 0, x_2 > 0\}$. Define $f : X \longrightarrow \mathbb{R}$, $g : X \times U \longrightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} 3x_1^2 + 2x_2^2, & 1 < x_1 \leq 4, 1 < x_2 \leq 4 \\ 100, & x_1 \geq 4 \quad \text{or } x_2 \geq 4 \quad \text{or } x_1 \geq 4, x_2 \geq 4 \\ 5, & \text{otherwise,} \end{cases}$$

$$g(x, u) = \begin{cases} x_2^2 - x_1^2 \sin u^i, & x_1 > 1, x_2 > 1 \\ 1 - \sin u^i, & \text{otherwise,} \end{cases}$$

where $u^i \in U = \{u^i|u^i = 2\pi + (i\pi)/2, i = 1, 2, \ldots\}$.

Let $H_{x^1,x^2} : [0, 1] \longrightarrow X$, defined as

$$H_{x^1,x^2}(\theta) = (((1-\theta)(x_1^1)^2 + \theta(x_1^2)^2)^{1/2}, ((1-\theta)(x_2^1)^2 + \theta(x_2^2)^2)^{1/2}),$$

where $x^1 = (x_1^1, x_2^1)$ and $x^2 = (x_1^2, x_2^2)$. Let $b_0 : X \times X \times [0, 1] \longrightarrow \mathbb{R}_+$ as

$$b_0(x^1, x^2, \theta) = \begin{cases} 0, & \text{if } x_1^1 \geq 4 \quad \text{or } x_2^1 \geq 4 \quad \text{or } x_1^1 \geq 4, x_2^1 \geq 4 \text{ and} \\ & x_1^2 < 4 \quad \text{or } x_2^2 < 4 \quad \text{or } x_1^2 < 4, x_2^2 < 4 \\ 1/\theta, & \text{if } x_1^2 \geq 4 \text{ or } x_2^2 \geq 4 \quad \text{or } x_1^2 \geq 4, x_2^2 \geq 4 \text{ and} \\ & x_1^1 < 4 \text{ or } x_2^1 < 4 \text{ or } x_1^1 < 4, x_2^1 < 4 \\ 1, & \text{otherwise} \end{cases}$$

and $b_1 : X \times X \times [0, 1] \longrightarrow \mathbb{R}_+$ as $b_1(x^1, x^2, \theta) = 1$.

Then, f is not CN, COCN, B$_0$-vex, or B$_0$-cave function, however, it is a B$_0$CN function with respect to the arc H_{x^1,x^2} and b_0, g is a B$_1$CN function for x with respect to the arc

H_{x^1,x^2} and b_1. They are nondifferentiable at $x^* = (1, 1)$, but they possess right differentials with respect to the arc $H_{x^*,x}$ for all $x \in X$, at $\theta = 0$, and they are given by

$$f^+(H_{x^*,x}(0)) = \begin{cases} 3(x_1)^2 + 2(x_2)^2 - 5, & \text{if both components of } H_{x^*,x} > 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$g^+(H_{x^*,x}(0), u^i) = \begin{cases} (x_2)^2 - (x_1)^2 - 1 + \sin u^i, & \text{if both components of} \\ & H_{x^*,x} > 1, u^i \in U, i = 1, 2, \dots \\ 0, & \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2)$.

The set of active constraints is given by $I(x^*) = \{i | g(x^*, u^i) = 0, u^i = \pi/2 + 2i\pi \in U, i = 1, 2, \dots\}$, we have

$$g^+(H_{x^*,x}(0), u^i) = \begin{cases} (x_2)^2 - (x_1)^2, & \text{if both components of } H_{x^*,x} > 1 \\ 0, & \text{otherwise,} \end{cases}$$

for $i \in I(x^*)$, where $x = (x_1, x_2)$.

For these functions $f(x)$ and $g(x, u^i)$, we consider the programming problem (P), and are easy to get that $\lambda_0^* = 1, \lambda_1^* = 1, \lambda_2^* = 1, \lambda_3^* = 1, \lambda_4^* = 0$, and $\lambda_5^* = 0, \dots$

$$\lambda_0^* f^+(H_{x^*,x}(0)) + \sum_{i \in I(x^*)} \lambda_i^* g^+(H_{x^*,x}(0), u^i) \geq 0, \quad \text{for all } x \in X,$$

which implies that $x^* = (1, 1)$ is an optimal solution of (P).

4 Duality

In this section, we consider the following Mond-Weir type dual problem (D) for problem (P):

$$\text{maximize } f(y)$$

$$\text{subject to } \lambda_0 f^+(H_{y,x}(0)) + \sum_{i \in \Delta} \lambda_i g^+(H_{y,x}(0), u^i) \geq 0, \tag{4.1}$$

$$\sum_{i \in \Delta} \lambda_i g(y, u^i) \geq 0, \tag{4.2}$$

$$\lambda_0 \geq 0, \ \ \lambda \geq 0, \lambda \in \Lambda. \tag{4.3}$$

Let $W = \{(y, u^i, \lambda_0, \lambda) | \lambda_0 f^+(H_{y,x}(0)) + \sum_{i \in \Delta} \lambda_i g^+(H_{y,x}(0), u^i) \geq 0, \sum_{i \in \Delta} \lambda_i g(y, u^i) \geq 0, \lambda_0 \geq 0, \lambda \geq 0, \lambda \in \Lambda\}$ be the set of feasible solutions of (D). The optimal values of (P) and (D) are denoted by $v(P)$ and $v(D)$, respectively.

Now, we establish duality relationship between problems (P) and (D).

Theorem 4.1 *(Weak Duality) Assume that x is feasible for (P) and $(y, u^i, \lambda_0, \lambda)$ is feasible for (D). If $f(x)$ is a DB_0CN at y and $\sum_{i \in \Delta} \lambda_i g(x, u^i)$ is a DB_1CN at y with respect to the same arc and*

$$b_0(x, y) = \lim_{\theta \to 0_+} b_0(x, y, \theta) > 0, \quad b_1(x, y) = \lim_{\theta \to 0_+} b_1(x, y, \theta),$$

then

$$f(y) \leq f(x).$$

Proof Suppose that

$$f(y) > f(x).$$

Since $f(x)$ is a DB_0CN at y, therefore using Theorem 2.1, we obtain

$$f^+(H_{y,x}(0)) \leq b_0(x,y)[f(x) - f(y)] < 0. \tag{4.4}$$

By the feasibility of x and $(y, u^i, \lambda_0, \lambda)$ for (P) and (D), respectively, we get

$$\sum_{i\in\Delta} \lambda_i g(x, u^i) \leq \sum_{i\in\Delta} \lambda_i g(y, u^i).$$

Now $\sum_{i\in\Delta} \lambda_i g(x, u^i)$ is a DB_1CN at y, using Theorem 2.1, we have

$$\sum_{i\in\Delta} \lambda_i g^+(H_{y,x}(0), u^i) \leq b_1(x,y)\left[\sum_{i\in\Delta} \lambda_i g(x, u^i) - \sum_{i\in\Delta} \lambda_i g(y, u^i)\right] \leq 0. \tag{4.5}$$

Letting $\lambda_0 > 0$, and adding (4.4) $\times\, \lambda_0$ and (4.5), we have

$$\lambda_0 f^+(H_{y,x}(0)) + \sum_{i\in\Delta} \lambda_i g^+(H_{y,x}(0), u^i) < 0$$

which is a contradiction to (4.1). Hence

$$f(y) \leq f(x).$$

□

Theorem 4.2 *(Strong Duality) Assume that x^* is an optimal solution of (P), and let $f^+(H_{x^*,x}(0))$ and $g^+(H_{x^*,x}(0), u^i)(i \in \Delta)$ be BCN functions of x, $g(x, u^i)(i \in J(x^*))$ be continuous at x^* with AC set $X \subseteq \mathbb{R}^n$. Then there exist $\lambda_0^* \geq 0, \lambda^* \geq 0, \lambda^* \in \Lambda$ such that $(x^*, u^i, \lambda_0^*, \lambda^*)$ is a feasible solution for (D), and the values of the objective functions for (P) and (D) are equal at x^*. Also if for every feasible solution $(y, u^i, \lambda_0, \lambda)$ for (D), f is a DB_0CN (or $DSTB_0CN$) at y and $\sum_{i\in\Delta} \lambda_i g(x, u^i)$ is a DB_1CN (or $DSTB_1CN$) at y for arbitrary $u^i \in U$ with*

$$b_0(x,y) = \lim_{\theta\to 0_+} b_0(x,y,\theta) > 0, \quad b_1(x,y) = \lim_{\theta\to 0_+} b_1(x,y,\theta),$$

then $(x^*, u^i, \lambda_0^*, \lambda^*)$ is an optimal solution for (D), and $v(P) = v(D)$.

Proof Since x^* is an optimal solution of (P), $f^+(H_{x^*,x}(0))$ and $g^+(H_{x^*,x}(0), u^i)(i \in \Delta)$ are BCN function of x, $g(x, u^i)(i \in J(x^*))$ is continuous at x^* with AC set $X \subseteq \mathbb{R}^n$, therefore by Theorem 3.1, there exist $\lambda_0^* \geq 0, \lambda^* \geq 0, \lambda^* \in \Lambda$, for $u^i \in U$ such that $(x^*, u^i, \lambda_0^*, \lambda^*)$ is a feasible solution of (D). Equality of the objective functions for (P) and (D) follows trivially.

If $(x^*, u^i, \lambda_0^*, \lambda^*)$ is not an optimal solution for (D), then exists $(y, u^i, \lambda_0, \lambda)$ feasible solution for (D) such that

$$f(x^*) < f(y)$$

which is a contradiction to Theorem 4.1 (Weak Duality). Hence $(x^*, u^i, \lambda_0^*, \lambda^*)$ is an optimal solution for (D) and, obviously, $v(P) = v(D)$. □

Theorem 4.3 *(Weak Duality) Assume that* x *is feasible for* (P) *and* $(y, u^i, \lambda_0, \lambda)$ *is feasible for* (D). *If* $f(x)$ *is a DSTB*$_0$*CN at* y *and* $\sum_{i\in\Delta}\lambda_i g(x, u^i)$ *is a DB*$_1$*CN (or DSTB*$_1$*CN) at* y *with respect to the same arc and*

$$b_0(x, y) = \lim_{\theta\to 0_+} b_0(x, y, \theta) > 0, \quad b_1(x, y) = \lim_{\theta\to 0_+} b_1(x, y, \theta),$$

then

$$f(y) \leq f(x).$$

The proof of Theorem 4.3 follows on the lines of Theorem 4.1.

Theorem 4.4 *(Strong Duality) Assume that* x^* *is an optimal solution of* (P), *and let* $f(x)$ *and* $g(x, u^i)(i \in \Delta)$ *be DBCN functions of* x *with respect to the same arc and* $\bar{b}(x, y) = \lim_{\theta\to 0_+} b(x, y, \theta) > 0$. *If there exists* $\hat{x} \in X^0$ *such that* $g(\hat{x}, u^i) < 0$, *then* $(x^*, u^i, \lambda_0^*, \lambda^*)$ *is an optimal solution for* (D), *and* $v(P) = v(D)$.

Proof Since $g(x, u^i)(i \in \Delta)$ is a DBCN function of x with respect to the same arc and $\bar{b}(x, y)$, hence we know $\sum_{i\in\Delta}\lambda_i g(x, u^i)$ is a DBCN function at y for arbitrary $u^i \in U$ and $\lambda \in \Lambda$ with respect to $\bar{b}(x, y)$, from assumption of the theorem and by Theorem 3.3, we know that there exist $\lambda_0^* \geq 0, \lambda^* \geq 0, \lambda^* \in \Lambda$, for $u^i \in U$ such that $(x^*, u^i, \lambda_0^*, \lambda^*)$ is a feasible solution of (D). Equality of the objective functions for (P) and (D) follows trivially.

If $(x^*, u^i, \lambda_0^*, \lambda^*)$ is not an optimal solution for (D), then exists $(y, u^i, \lambda_0, \lambda)$ feasible solution for (D) such that

$$f(x^*) < f(y)$$

which is a contradiction to Theorem 4.1 (Weak Duality). Hence $(x^*, u^i, \lambda_0^*, \lambda^*)$ is an optimal solution for (D) and, obviously, $v(P) = v(D)$. □

Theorem 4.5 *(Strong Duality) Assume that* x^* *is an optimal solution of* (P), *and let* $f(x)$ *and* $g(x, u^i)(i \in \Delta)$ *be DSTBCN functions of* x *with respect to the same arc and* $\bar{b}(x, y) = \lim_{\theta\to 0_+} b(x, y, \theta) > 0$. *If there exists* $\hat{x} \in X^0$ *such that* $g(\hat{x}, u^i) < 0$, *then* $(x^*, u^i, \lambda_0^*, \lambda^*)$ *is an optimal solution for* (D), *and* $v(P) = v(D)$.

The proof of Theorem 4.5 follows on the lines of Theorem 4.4.

Example 4.1 we consider the dual problem (D) of the primal programming in Example 3.1 and know $x^* = (1, 1)$ is an optimal solution of primal programming. For $\lambda_0^* = 1, \lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*, \lambda_5^*, \ldots) = (1, 1, 1, 0, 0, \ldots)$, $u^i = \pi/2 + 2i\pi \in U, i \in \triangle$, the conditions of Theorem 4.2 are satisfied. Hence, $(x^*, u^i, \lambda_0^*, \lambda^*)$ is an optimal solution of this dual programming and $v(P) = v(D)$.

5 Conclusions and outlook

The concepts of B-arcwise connected (BCN) and strictly B-arcwise connected functions (STBCN) based on arcwise connected functions (CN) and B-vex functions are introduced into the paper. Some differential and extreme properties are studied. The optimality conditions and Mond-Weir type duality results are obtained for a nonlinear constrained semi-infinite programming problem involving BCN and STBCN functions. The BCN function will play a vital role in many aspects of mathematical programming including optimality conditions and duality theorems, which will be used in constrained multiobjective programming, generalized convex programming, and fractional programming. There are many practical problems which may be treated by SIP techniques, including engineering design [17], orthogonal wavelet filter design [18], reliability testing [18], robot trajectory planning [9], air pollution control [9], constrained Chebyshev approximation and new product development planning etc. The arcwise connected SIP in this paper will also be applied in many practical problems.

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